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The Corona Theorem and the Canonical Factorization of Triangular AP Matrix Functions—Effective Criteria and Explicit Formulas*

M. A. Bastos, Yu. I. Karlovich, A. F. dos Santos

and

P. M. Tishin

Hydroacoustic Department, Marine Hydrophysical Institute, Ukrainian Academy of Sciences, Preobrazhenskaya Str. 3, 270100 Odessa, Ukraine

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Using results of the companion paper, the authors obtain effective criteria for the existence of canonical factorization for new classes of triangular almost periodic matrix functions of the form $\begin{bmatrix} e^{i\lambda\xi} & 0 \\ f(\xi) & e^{-i\lambda\xi} \end{bmatrix}$. The important role in the proof of necessity in these criteria is played by a result on the stability of canonical AP factorizability obtained in the present paper. By finding corona solutions for a pair of binomial almost periodic functions, it is possible to calculate explicitly the factors of a canonical factorization for a class of considered matrix functions with trinomial f. © 1998 Academic Press

Key Words: canonical factorization; corona theorem; almost periodic matrix function.

1. INTRODUCTION

This paper is a continuation of and companion to our previous paper [1]. To avoid repetition we will use definitions, notations, and results of [1] without additional comments.

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In this paper we derive effective criteria and explicit formulas for canonical factorization of triangular almost periodic (AP) matrix-valued functions. Having obtained, in the companion paper [1], a criterion for the existence of canonical factorization for AP_W matrix-valued functions of the form

$$G(\xi) = \begin{bmatrix} e^{i\lambda\xi} & 0 \\ f(\xi) & e^{-i\lambda\xi} \end{bmatrix} \tag{1.1}$$

and formulas for factorization factors in terms of corona solutions, we turn now to the question of calculating the factors and getting canonical factorization criteria for concrete classes of triangular AP_W matrix-valued functions in terms of their entries. The factorization problem for different classes of triangular AP_W matrix functions was studied in [10, 12, 13, 9, 2, 7, 16]. For applications of these results to the invertibility study of difference operators on a semi-axis or on a finite interval and to Fredholm theory of singular integral operators with semi-almost periodic matrix coefficients and of convolution type operators with semi-almost periodic matrix symbols, see, e.g., [10, 8, 11, 3, 9].

The paper is organized as follows. In Section 2 we state a result to be used later, about the stability of canonical AP_W factorizability of AP_W matrix-valued functions under small perturbations of their coefficients and exponents. In Section 3, for the periodic and AP_W versions of the corona theorem, some resutls on the determination of the corona solutions are presented. In Sections 4–6 the canonical AP factorization for concrete classes of AP_W matrix-valued functions is studied. A criterion for canonical AP factorization is given in Section 4 for a class of periodic functions and in Section 5 for classes of AP_W matrix-valued functions defined by (1.1) with f a quotient of polynomials. Finally, in Section 6, by applying the corona theorem we prove in an essentially simpler way a criterion for AP factorization for the matrix function (1.1) in the case of a trinomial f that was already established in [12]. Moreover, by using corona solutions obtained in Section 3 we calculate explicitly the factors of this factorization. We note that the factors are obtained here for the first time.

2. STABILITY OF AP_W FACTORIZABILITY

Let AP_W be the algebra of absolutely convergent Fourier series $\sum \hat{f}(\lambda)e^{i\lambda\xi}$, where λ runs through the spectrum $\Omega(f) \subset \mathbb{R}$ of f, and

$$AP_W^{\pm} = \left\{ f \in AP_W : \hat{f}(\lambda) = 0 \text{ for } \pm\lambda < 0 \right\}.$$

We will use notations $[X]_{n,n}$ for the class of $n \times n$ matrix functions with entries in X, and $\|G\|_W = \sum_\lambda \|G_\lambda\|$ if $G = \sum G_\lambda e_\lambda \in [\mathrm{AP}_W]_{n,n}$, where G_λ are constant complex matrices.

Below we will use the following result on the stability of the canonical AP_W factorizability of AP_W matrix-valued functions under small perturbations of coefficients and exponents (cf. [18, Theorem 2] and [9, Theorem 5.6]; see also [13, Lemma 3.1 and Corollary 3.1]).

Let a matrix-valued function $G \in [\mathrm{AP}_W]_{n,n}$ be represented in the form

$$G(\xi) = \sum_h G_h e^{ih\xi}, \qquad \xi \in \mathbb{R},$$

where G_h are constant complex matrices and $h \in \mathbb{R}$ runs through an Abelian group $H = H(G)$ generated by an at most countable set Γ_G of real numbers γ_j which are linearly independent over the ring $\mathbb{Z}$.

Let us associate to each $\gamma_j \in \Gamma_G$ a real number γ_j' and consider the Abelian group H' generated by the set $\Gamma_G' = \{\gamma_j' \colon \gamma_j \in \Gamma_G\}$. The map $\nu\colon \Gamma_G \to \Gamma_G'$, $\gamma_j \mapsto \gamma_j'$, can be extended to a homomorphism $\nu\colon H \to H'$ that induces a Banach algebra homomorphism

$$\psi_\nu\colon [\mathrm{AP}_W(H)]_{n,n} \to [\mathrm{AP}_W(H')]_{n,n},$$
$$\sum_{h \in H} G_h e^{ih\xi} \mapsto \sum_{h \in H} G_h e^{i\nu(h)\xi},$$

where $[\mathrm{AP}_W(H)]_{n,n} = \{G \in [\mathrm{AP}_W]_{n,n} \colon \Omega(G) \subset H\}$.

THEOREM 2.1. *Let G be a matrix-valued function in $[AP_W]_{n,n}$,*

$$G(\xi) = \sum_{h \in \Omega(G)} G_h e^{ih\xi},$$

and let G have a canonical AP_W factorization. Then for each $h \in \Omega(G)$ and for each $\gamma_j \in \Gamma_G$, there is a neighborhood $U(G_h)$ of G_h in $\mathbb{C}^{n\times n}$ and a neighborhood U_{γ_j} of γ_j in $\mathbb{R}$ such that, for every $G_h' \in U(G_h)$ and for every $\gamma_j' \in U_{\gamma_j}$, the matrix-valued function

$$F(\xi) = \sum_{h \in \Omega(G)} G_h' e^{i\nu(h)\xi}$$

has a canonical AP_W factorization.

Proof. Let $G \in [\mathrm{AP}_W]_{n,n}$ have a canonical AP_W factorization

$$G = G_+ G_-, \tag{2.1}$$

with $G_+^{\pm 1} \in [\mathrm{AP}_W^+]_{n,n}$, $G_-^{\pm 1} \in [\mathrm{AP}_W^-]_{n,n}$, and $M(G_+^{\pm 1}) = I$, where I stands for the identity matrix.

First, we show that

$$\Omega\left(G_{\pm}^{\pm 1}\right) \subset H. \tag{2.2}$$

Let

$$G_{+}^{-1} = B_{+} + C_{+}, \qquad G_{-} = B_{-} + C_{-}, \tag{2.3}$$

where $B_{\pm}, C_{\pm} \in [\mathrm{AP}_W^{\pm}]_{n,n}$, $\Omega(B_{\pm}) \subset H$, $\Omega(C_{\pm}) \cap H = \emptyset$, and $M(B_{+}) = I$.

Since $\Omega(G^{\pm 1}) \subset H$, we get from $G_{+}^{-1} G = G_{-}$ and (2.3) that $B_{+} G = B_{-}$, whence by (2.1),

$$B_{+} G_{+} = B_{-} G_{-}^{-1}. \tag{2.4}$$

Since $B_{+} G_{+} \in [\mathrm{AP}_W^{+}]$, $B_{-} G_{-}^{-1} \in [\mathrm{AP}_W^{-}]_{n,n}$, and $M(B_{+} G_{+}) = I$, the equality (2.4) yields

$$B_{+} G_{+} = B_{-} G_{-}^{-1} = I.$$

Then $G_{+}^{-1} = B_{+}$, $G_{-} = B_{-}$, and, hence, $\Omega(G_{+}^{-1}), \Omega(G_{-}) \subset H$. Analogously we may conclude that $\Omega(G_{+}), \Omega(G_{-}^{-1}) \subset H$.

As we have shown, there is an AP_W factorization (2.1) satisfying (2.2). Then there is a sequence of matrix almost periodic polynomials $P_k^{\pm}$ such that

$$\lim_{k \to \infty} \|P_k^{\pm} - G_{\pm}^{-1}\|_W = 0$$

and $\Omega(P_k^{\pm}) \subset H$. Then we may choose $k \in \mathbb{N}$ such that

$$\|I - P_k^{+} G P_k^{-}\|_W < \tfrac{1}{2}. \tag{2.5}$$

Fix this k and choose neighborhoods $U(G_h)$ of coefficients G_h, $h \in \Omega(G)$, such that

$$\left\| P_k^{+} \left(G - \sum_{h \in \Omega(G)} G_h' e^{ih\xi} \right) P_k^{-} \right\|_W < \tfrac{1}{2} \tag{2.6}$$

for all $G_h' \in U(G_h)$ and all $h \in \Omega(G)$. From (2.5) and (2.6) we have

$$\left\| I - P_k^{+} \sum_{h \in \Omega(G)} G_h' e^{ih\xi} P_k^{-} \right\|_W < 1. \tag{2.7}$$

Choose now neighborhoods U_{γ_j} of generators $\gamma_j \in \Gamma_G$ of the group H such that, for all $\gamma_j' \in U_{\gamma_j}$,

$$\psi_\nu\left(P_k^{\pm}\right) \in \left[\mathrm{AP}_W^{\pm}\right]_{n,n}. \tag{2.8}$$

This is possible since $\Omega(P_k^{\pm}) \subset H \cap \mathbb{R}_{\pm}$ and the sets $\Omega(P_k^{\pm})$ are finite. Moreover, since $(P_k^{\pm})^{-1} \in [\mathrm{AP}_W^{\pm}]_{n,n}$, we get that $0 \in \Omega((P_k^{\pm})^{-1})$ and the sets $\Omega((P_k^{\pm})^{-1}) \setminus \{0\} \subset \mathbb{R}_{\pm}$ are separated from zero. Hence

$$\left(\psi_\nu\left(P_k^{\pm}\right)\right)^{-1} = \psi_\nu\left(\left(P_k^{\pm}\right)^{-1}\right) \in \left[\mathrm{AP}_W^{\pm}\right]_{n,n}. \tag{2.9}$$

Since ψ_ν is a homomorphism of $[\mathrm{AP}_W(H)]_{n,n}$ into $[\mathrm{AP}_W(H')]_{n,n}$, where $H' = \nu(H)$ and since $\|\psi_\nu(G)\|_W \leq \|G\|_W$, we get from (2.7) that

$$\begin{aligned} &\left\| I - \psi_\nu(P_k^+) \sum_{h \in \Omega(G)} G_h' e^{i\nu(h)\xi} \psi_\nu(P_k^-) \right\|_W \\ &\qquad \leq \left\| I - P_k^+ \sum_{h \in \Omega(G)} G_n' e^{ih\xi} P_k^- \right\|_W < 1. \end{aligned} \tag{2.10}$$

By Theorem 1.1 in [12], from (2.10) it follows that the matrix-valued function

$$F_1 = \psi_\nu(P_k^+) \sum_{h \in \Omega(G)} G_h' e^{i\nu(h)\xi} \psi_\nu(P_k^-)$$

has a canonical AP_W factorization $F_1 = F_1^+ F_1^-$, where $\Omega(F_1^{\pm})$, $\Omega((F_1^{\pm})^{-1}) \subset H'$ because $\Omega(F_1) \subset H'$. Then, by virtue of (2.8) and (2.9), the matrix-valued function

$$F(\xi) = \sum_{h \in \Omega(G)} G_h' e^{i\nu(h)\xi}$$

has a canonical AP_W factorization $F = F_+ F_-$ with factors

$$F_+ = \left(\psi_\nu(P_k^+)\right)^{-1} F_1^+, \qquad F_- = F_1^- \left(\psi_\nu(P_k^-)\right)^{-1}.$$

∎

3. CORONA SOLUTIONS

Let H_∞^+ be the Hardy space of bounded analytic functions on $\Pi^+ = \{z\colon \operatorname{Im} z > 0\}$, and let $f_1, \ldots, f_n$ be functions in H_∞^+. By the corona theorem (see, e.g., [6]), the corona problem

$$\sum_{j=1}^{n} f_j \tilde{f}_j = 1 \tag{3.1}$$

has solutions $\tilde{f}_1, \ldots, \tilde{f}_n$ in H_∞^+ if and only if

$$\inf_{z \in \Pi^+} \max_{j=1,\ldots,n} \left| f_j(z) \right| > 0.$$

Corona problem (3.1) in the class $AP^{+} \subset H_{\infty}^{+}$ is studied in [19] (see also [17] for its AP_W version).

If functions $f_1, \dots, f_n \in AP^{+}$ are periodic polynomials with period 2π, then the substitution $t = e^{i\xi}$ leads to usual polynomials

$$\hat{f}_j(t) = f_j(\arg t), \qquad j = 1, \dots, n,$$

on the unit circle $\Gamma = \{z \colon |z| = 1\}$, for which we get a version of the corona theorem for $H_\infty(D)$, where D is the unit disk $D = \{z \in \mathbb{C} \colon |z| < 1\}$. In this case, if the data are polynomials, we make use of the following theorem.

THE BEZOUT THEOREM (see, e.g., [4]). *Let f, g be polynomials that do not have common roots. Then there exist polynomials $\tilde{f}, \tilde{g}$ such that*

$$f\tilde{f} + g\tilde{g} = 1. \tag{3.2}$$

In this section we present, for this periodic version, some useful results about the determination of the corona solutions.

Given polynomials

$$f(t) = v_m \prod_{i=1}^{m} (t - c_i) = v_0 + v_1 t + \cdots + v_m t^m, \tag{3.3}$$

$$g(t) = w_n \prod_{j=1}^{n} (t - s_j) = w_0 + w_1 t + \cdots + w_n t^n, \tag{3.4}$$

we look for polynomials

$$\tilde{f}(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}, \tag{3.5}$$

$$\tilde{g}(t) = b_0 + b_1 t + \cdots + b_{m-1} t^{m-1}, \tag{3.6}$$

satisfying the corona condition (3.1). To this end we can solve the algebraic system

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\hphantom{v_m \quad v_m \quad \ddots \quad \ddots \quad v_m}}^{n} & \overbrace{\hphantom{w_n \quad w_n \quad \ddots \quad w_n}}^{m} \end{array} \\ m+n \left\{ \vphantom{\begin{array}{c} 1\\1\\1\\1\\1\\1\\1\\1\\1 \end{array}} \right. & \left[\begin{array}{ccccc|cccc} v_m & & & & 0 & w_n & & & 0 \\ v_{m-1} & v_m & & & & w_{n-1} & w_n & & \\ \vdots & v_{m-1} & \ddots & & & \vdots & w_{n-1} & \ddots & \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & w_n \\ v_0 & \vdots & & \ddots & v_m & \vdots & \vdots & & w_{n-1} \\ & v_0 & & & v_{m-1} & w_0 & \vdots & & \vdots \\ & & \ddots & & \vdots & & w_0 & & \vdots \\ & & & \ddots & \vdots & & & \ddots & \vdots \\ 0 & & & & v_0 & 0 & & & w_0 \end{array} \right] \end{array} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ \vdots \\ a_0 \\ b_{m-1} \\ b_{m-2} \\ \vdots \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3.7}$$

The determinant of this system is called the resultant of polynomials f, g and is denoted by $\operatorname{Res}(f, g)$. As it is known (see, e.g., [15], Chap. 4, Proposition 8.3),

$$\operatorname{Res}(f, g) = v_m w_n \prod_{i=1}^{m} \prod_{j=1}^{n} (c_i - s_j). \tag{3.8}$$

Then we arrive at the following sharpening of the Bezout theorem.

PROPOSITION 3.1. *Let f, g be polynomials given by* (3.3) *and* (3.4) *with* $v_m w_n \neq 0$. *The corona problem* (3.2) *has a solution $\tilde{f}, \tilde{g}$ of the form* (3.5) *and* (3.6) *if and only if*

$$c_i \neq s_j \quad \text{for all } i = 1, \dots, m \text{ and all } j = 1, \dots, n. \tag{3.9}$$

If (3.9) *holds, then the coefficients of polynomials $\tilde{f}, \tilde{g}$ can be found from* (3.7) *by the Cramér rule.*

We emphasize that (3.9) is a criterion for the solvability of the corona problem (3.2) in the class of polynomials of arbitrary degree. If we look for $\tilde{f}, \tilde{g}$ in the class of bounded analytic functions on the unit disk D, condition (3.9) must be fulfilled only for the zeros of f, g which lie in D.

Now, for polynomial corona data f, g of some particular form, we construct, explicitly, corona solutions which will be used later. Naturally, their expressions look simpler than in the general case (cf. Proposition 3.1).

PROPOSITION 3.2. *For the corona data*

$$f(t) = t^m - c, \qquad g(t) = t^n - s, \qquad t \in \Gamma,$$

where $c, s \in \mathbb{C}$; $m, n \in \mathbb{N}$, $m < n$ and m, n are relatively prime, the corona problem (3.2) *is solvable in the class of polynomials if and only if $c^n \neq s^m$.*

If the latter holds, then one pair of corona solutions is

$$\tilde{f}(t) = -\frac{1}{c^n - s^m} \sum_{j=0}^{m-1} \sum_{k=k_{j-1}+1}^{k_j} c^{n-1-k} s^j t^{km-jn},$$

$$\tilde{g}(t) = \frac{1}{c^n - s^m} \sum_{j=0}^{m-1} c^{n-1-k_j} s^j t^{(k_j+1)m-(j+1)n},$$

where

$$k_{-1} = -1, \qquad k_j = E((j+1)n/m), \qquad j = 0, \dots, m-2;$$

$$k_{m-1} = n - 1,$$

and $E(\xi)$ denotes the integer part of $\xi \in \mathbb{R}$.

Proof. For given polynomials f, g we get from (3.7) or (3.8) that

$$\operatorname{Res}(f, g) = (-1)^{n(m+1)} c^n + (-1)^m s^m = (-1)^{n(m+1)} (c^n - s^m).$$

Then by Proposition 3.1, for these polynomials f, g the corona problem (3.2) is solvable in the class of polynomials if and only if $c^n \neq s^m$. Taking into account that $(k_{m-1} + 1)m = mn$ we get

$$\begin{aligned} &(t^m - c)\tilde{f}(t) \\ &\quad = \frac{-1}{c^n - s^m} \sum_{j=0}^{m-1} \left(-c^{n-1-k_{j-1}} s^j t^{(k_{j-1}+1)m - jn} + c^{n-1-k_j} s^j t^{(k_j+1)m-jn} \right), \\ &(t^n - s)\tilde{g}(t) \\ &\quad = \frac{1}{c^n - s^m} \sum_{j=0}^{m-1} \left(c^{n-1-k_j} s^j t^{(k_j+1)m-jn} - c^{n-1-k_j} s^{j+1} t^{(k_j+1)m-(j+1)n} \right), \end{aligned}$$

whence

$$(t^m - c)\tilde{f}(t) + (t^n - s)\tilde{g}(t) = 1.$$

∎

We generalize Proposition 3.2 to the class of AP_W functions with bounded spectra. Let $E(\xi)$ be the integer part of $\xi \in \mathbb{R}$ and let $\tilde{E}(\xi)$ be the greatest integer number which is less than ξ.

LEMMA 3.3. *Let*

$$f_+(\xi) = e^{i\mu\xi} - c, \qquad g_+(\xi) = e^{i\nu\xi} - s,$$

where $c, s \in \mathbb{C}$, $0 < \mu \leq \nu$. Then for the solvability of the corona problem $f_+\tilde{f}_+ + g_+\tilde{g}_+ = 1$ in the class $AP_W^+(\nu) = \{f \in AP_W\colon \Omega(f) \subset [0, \nu]\}$ it is necessary and sufficient that

$$\begin{aligned} c^n \neq s^m, &\quad \textit{if } \nu/\mu \textit{ is rational with } m, n \in \mathbb{N} \textit{ relatively prime such that} \\ &\quad \nu/\mu = m/n; \\ |c|^\nu \neq |s|^\mu, &\quad \textit{if } \nu/\mu \textit{ is irrational}. \end{aligned}$$

If these conditions hold, then one pair of corona solutions $\tilde{f}_+, \tilde{g}_+ \in AP_W^+(\nu)$ *which possess the additional property* $\Omega(\tilde{f}_+) \subset [0, \nu[$, $\Omega(\tilde{g}_+) \subset [0, \mu[$, *has, respectively, the form*:

(i)

$$\tilde{f}_+(\xi) = -\frac{1}{c^n - s^m} \sum_{j=0}^{m-1} \sum_{k=k_{j-1}+1}^{k_j} c^{n-k-1} s^j \exp(i(k\mu - j\nu)\xi),$$

$$\tilde{g}_+(\xi) = \frac{1}{c^n - s^m} \sum_{j=0}^{m-1} c^{n-k_j-1} s^j \exp\Big(i\big((k_j+1)\mu - (j+1)\nu\big)\xi\Big)$$

with $k_j = \tilde{E}((j+1)\nu/\mu)$, *if* $\nu/\mu = n/m$ *is rational*;

(ii) $$\tilde{f}_+(\xi) = -\sum_{j=0}^{\infty} \sum_{k=k_{j-1}+1}^{k_j} c^{-k-1} s^j \exp(i(k\mu - j\nu)\xi),$$

$$\tilde{g}_+(\xi) = \sum_{j=0}^{\infty} c^{-k_j-1} s^j \exp\Big(i\big((k_j+1)\mu - (j+1)\nu\big)\xi\Big)$$

with $k_j = \tilde{E}((j+1)\nu/\mu)$ *and* $0 \notin \Omega(\tilde{g}_+)$, *if* ν/μ *is irrational and* $|c|^\nu > |s|^\mu$;

(iii) $$\tilde{f}_+(\xi) = \sum_{j=1}^{\infty} \sum_{\tau=\tau_j+1}^{\tau_{j+1}} c^{\tau-1} s^{-j} \exp(i(-\tau\mu + j\nu)\xi),$$

$$\tilde{g}_+(\xi) = -\sum_{j=1}^{\infty} c^{\tau_j} s^{-j} \exp\Big(i\big(-\tau_j\mu + (j-1)\nu\big)\xi\Big)$$

with $\tau_j = E((j-1)\nu/\mu)$ *and* $0 \notin \Omega(\tilde{f}_+)$, *if* ν/μ *is irrational and* $|c|^\nu < |s|^\mu$.

Proof. Since $0 < \mu \le \nu$, the corona data f_+, g_+ as well as the desired corona solutions $\tilde{f}_+, \tilde{g}_+$ belong to $\mathrm{AP}_W^+(\nu)$. Then automatically $\tilde{g}_+ \in \mathrm{AP}_W^+(\mu)$ because $f_+ \in \mathrm{AP}_W^+(\mu)$. By Theorem 5.5 from [1], the corona problem $f_+\tilde{f}_+ + g_+\tilde{g}_+ = 1$ is solvable in $\mathrm{AP}_W^+(\nu)$ if and only if

$$\inf_{z\in\mathbb{C}} \max\big\{|f_+(z)|, |g_+(z)|\big\} > 0. \tag{3.10}$$

Obviously, (3.10) holds if the preimages $f_+^{-1}(D_\varepsilon(c))$ and $g_+^{-1}(D_\varepsilon(s))$ of the discs $D_\varepsilon(c) = \{z\colon |z-c| < \varepsilon\}$ and $D_\varepsilon(s) = \{z\colon |z-s| < \varepsilon\}$ do not intersect for a sufficiently small $\varepsilon > 0$.

Investigate the fulfillment of (3.10) by comparing the zeros of the functions

$$f_+(z) = e^{i\mu z} - c, \qquad g_+(z) = e^{i\nu z} - s.$$

The zeros of f_+ and g_+ are given by

$$z = (\arg c + 2\pi k - i \log|c|)/\mu, \qquad k \in \mathbb{Z}, \tag{3.11}$$

and

$$z = (\arg s + 2\pi r - i \log|s|)/\nu, \qquad r \in \mathbb{Z}. \tag{3.12}$$

Then $f_+^{-1}(D_\varepsilon(c))$ and $g_+^{-1}(D_\varepsilon(s))$ are contained in the sets

$$\mathcal{F}_\varepsilon(c) = \bigcup_{k\in\mathbb{Z}} \left(\frac{1}{\mu} Y_\varepsilon(c) + \frac{2\pi k}{\mu} \right) \quad \text{and} \quad G_\varepsilon(s) = \bigcup_{k\in\mathbb{Z}} \left(\frac{1}{\nu} Y_\varepsilon(s) + \frac{2\pi k}{\nu} \right),$$

respectively, where

$$Y_\varepsilon(a) = \Big\{ z\colon \min_{\varphi\in[0,2\pi)} \arg(a + \varepsilon e^{i\varphi}) < \operatorname{Re} z < \max_{\varphi\in[0,2\pi)} \arg(a + \varepsilon e^{i\varphi}),$$
$$-\log(|a| + \varepsilon) < \operatorname{Im} z < -\log(\max\{|a| - \varepsilon, 0\}) \Big\}.$$

If the imaginary parts of the roots in (3.11) and (3.12) are different, i.e., if $|c|^{1/\mu} \neq |s|^{1/\nu}$ or, equivalently,

$$|c|^\nu \neq |s|^\mu,$$

then $\mathcal{F}_\varepsilon(c) \cap G_\varepsilon(s) = \varnothing$ for a small $\varepsilon > 0$ and hence (3.10) holds.

If $|c|^\nu = |s|^\mu$ but the number ν/μ is rational, then the condition

$$c^n \neq s^m, \tag{3.13}$$

where n, m are co-prime numbers in $\mathbb{N}$ such that $\nu/\mu = n/m$, denotes that

$$(\arg c + 2\pi k)/\mu \neq (\arg s + 2\pi r)/\nu \quad \text{for all } k, r \in \mathbb{Z}.$$

Consequently, (3.13) implies that the number $(\arg c)/\mu - (\arg s)/\nu$ does not lie on the grid $\{2\pi l/(\nu m)\colon l \in \mathbb{Z}\}$ whence

$$\inf_{k,r\in\mathbb{Z}} |(\arg c + 2\pi k)/\mu - (\arg s + 2\pi r)/\nu| > 0$$

and again (3.10) holds.

It is easily seen that (3.10) is violated in other cases.

Let us show that the formulas (i), (ii), and (iii) are corona solutions.

In case (i) the formulas for $\tilde{f}_+, \tilde{g}_+$ follow from Proposition 3.2 under the replacement $t = \exp(i\nu\xi/n)$.

In cases (ii) and (iii) all the expressions for $\tilde{f}_+, \tilde{g}_+$ are well defined. It is easily seen that the series in case (ii) (respectively in case (iii)) converges due to the inequality $|c|^\nu > |s|^\mu$ (resp. $|c|^\nu < |s|^\mu$).

Let ν/μ be irrational and $|c|^{\nu} > |s|^{\mu}$. Then

$$f_+(\xi)\tilde{f}_+(\xi) = \sum_{j=0}^{\infty} c^{-k_{j-1}-1} s^j \exp\Big(i\big((k_{j-1}+1)\mu - j\nu\big)\xi\Big)$$

$$- \sum_{j=0}^{\infty} c^{-k_j-1} s^j \exp\Big(i\big((k_j+1)\mu - j\nu\big)\xi\Big),$$

$$g_+(\xi)\tilde{g}_+(\xi) = -\sum_{j=0}^{\infty} c^{-k_j-1} s^{j+1} \exp\Big(i\big((k_j+1)\mu - (j+1)\nu\big)\xi\Big)$$

$$+ \sum_{j=0}^{\infty} c^{-k_j-1} \exp\Big(i\big((k_j+1)\mu - j\nu\big)\xi\Big),$$

whence, because $k_{-1} = \tilde{E}(0) = -1$,

$$f_+(\xi)\tilde{f}_+(\xi) + g_+(\xi)\tilde{g}_+(\xi) = c^{-k_{-1}-1}\exp\big(i(k_{-1}+1)\mu\xi\big) = 1.$$

Obviously, $0 \notin \Omega(\tilde{g}_+)$.

(iii) is considered analogously. Notice also that in this case the functions $\tilde{f}_+, \tilde{g}_+$ can be rewritten in the form

$$\tilde{f}_+(\xi) = \sum_{j=-\infty}^{-1} \sum_{k=k_{j-1}+1}^{k_j} c^{-k-1} s^j \exp\big(i(k\mu - j\nu)\xi\big),$$

$$\tilde{g}_+(\xi) = -\sum_{j=-\infty}^{-1} c^{-k_j-1} s^j \exp\Big(i\big((k_j+1)\mu - (j+1)\nu\big)\xi\Big)$$

in view of the relation

$$E((j-1)\nu/\mu) = -\tilde{E}((-j+1)\nu/\mu) - 1,$$

i.e., $\tau_j = -k_{-j} - 1$.

Finally, $\Omega(\tilde{f}_+) \subset [0, \nu[$, $\Omega(\tilde{g}_+) \subset [0, \mu[$ since $k_j\mu < (j+1)\nu \leq (k_j+1)\mu$ and hence

$$0 \leq \big(k_{j-1}+1\big)\mu - j\nu \leq k_j\mu - j\nu = k_j\mu - (j+1)\nu + \nu < \nu,$$

$$0 \leq \big(k_j+1\big)\mu - (j+1)\nu = k_j\mu - (j+1)\nu + \mu < \mu.$$

∎

Notice that in view of Theorem 5.5 in [1], (3.10) is fulfilled automatically in cases (i)–(iii) of Lemma 3.3 due to the explicit solutions produced.

4. PERIODIC MATRIX FUNCTIONS

Let G be defined by

$$G(t) = \begin{bmatrix} t^n & 0 \\ f_+(t) \prod_{k=1}^{n} f_k^-(t) & t^{-n} \end{bmatrix}, \qquad |t| = 1, \tag{4.1}$$

where

$$f_+(t) = \sum_{i=0}^{m} a_i t^i, \qquad f_k^-(t) = \sum_{i_k=0}^{m+n} s_k^{i_k} t^{-i_k}, \tag{4.2}$$

and $n > 0$, $m > 0$.

The matrix-valued function G defined by (4.1) with $t = e^{i\xi}$ is a particular case of (1.1), and in this section we get a criteria for canonical AP factorization for this class of periodic matrix functions. We emphasize that the criteria stated in Theorems 5.3 and 5.6 of [1] can give only sufficient conditions for the canonical AP factorizability of the matrix function (4.1) if the representation of the nondiagonal entry f of (4.1) in the form

$$f = f_+ f_- - h_- e_{-\lambda} \tag{4.3}$$

is fixed. Notice that such representation is not unique.

LEMMA 4.1. *Let G be defined by* (4.1) *and* (4.2) *with $t = e^{i\xi}$ and*

$$|s_k| < 1, \qquad k = 1, \dots, n. \tag{4.4}$$

Then the following two assertions are equivalent:

(i) *G has a canonical AP factorization.*

(ii) *The roots of the polynomial f_+ in* (4.2) *do not belong to the set* $\{s_k\}_{k=1}^{n}$.

Proof. Assume that condition (ii) is satisfied. Let

$$f(t) = f_+(t) \prod_{k=1}^{n} f_k^-(t) \tag{4.5}$$

and

$$f_-(t) = \prod_{k=1}^{n} \left(1 - s_k t^{-1}\right)^{-1}. \tag{4.6}$$

Then we get

$$f(t) + h_-(t) t^{-n} = f_+(t) f_-(t), \tag{4.7}$$

where

$$h_-(t) = f_+(t)\sum_{j=1}^{n}\left(\prod_{k=1}^{j-1} f_k^-(t)\right)\left(s_j^{m+n}t^{-m}\sum_{i_j=0}^{\infty} s_j^{i_j}t^{-i_j}\right)\left(\prod_{k=j+1}^{n}\left(1-s_k t^{-1}\right)^{-1}\right).$$

Define

$$g_+(t) = t^n f_-^{-1}(t) = \prod_{k=1}^{n}(t-s_k). \tag{4.8}$$

We have

$$f_+ \in \mathrm{AP}_W^+, \qquad g_+ \in \mathrm{AP}_W^+, \qquad f_-^{\pm 1} \in \mathrm{AP}_W^-, \qquad h_- \in \mathrm{AP}_W^-.$$

The polynomials f_+ and g_+ do not have common roots and by the Bezout theorem there exist polynomials $\tilde{f}_+, \tilde{g}_+$ of $t = e^{i\xi}$ such that $f_+\tilde{f}_+ + g_+\tilde{g}_+ = 1$. Then, by Theorem 4.2 from [1], the matrix-valued function G admits a canonical AP factorization with periodic factors $G_\pm$ (see formulas (3.10) and (3.11) in [1]).

Conversely, assume that (i) is satisfied and let us show (ii).

Suppose the polynomials f_+ and g_+ have common roots and $\mathscr{S}$ is the set of common roots of f_+ and g_+. Denote by n_0 the cardinality of $\mathscr{S}$.

Let

$$p(t) = \prod_{s\in\mathscr{S}}(t-s). \tag{4.9}$$

The polynomials

$$f_1^+ = f_+/p, \qquad g_1^+ = g_+/p \tag{4.10}$$

do not have common roots, and by the Bezout theorem, there are polynomials $\tilde{f}_1^+$ and $\tilde{g}_1^+$ such that $f_1^+\tilde{f}_1^+ + g_1^+\tilde{g}_1^+ = 1$.

We have from (4.8) and (4.10) that

$$f_+(t)f_-(t) = \frac{f_+(t)}{g_+(t)}t^n = \frac{f_1^+(t)}{g_1^+(t)}t^n. \tag{4.11}$$

In view of (4.5), (4.7), and (4.11) the matrix function

$$G_1(t) = \begin{bmatrix} t^n & 0 \\ f_1^+(t)t^n/g_1^+(t) & t^{-n} \end{bmatrix}$$

has a canonical AP factorization simultaneously with (4.1).

Let

$$g_-(t) = g_1^+(t)t^{-n+n_0} = g_+(t)t^{-n}/(p(t)t^{-n_0}).$$

Since p is a polynomial of degree $n_0 > 0$, we get from (4.8), (4.9), and (4.4) that $g_-^{\pm 1} \in \mathrm{AP}^-$.

Then the matrix function

$$G_2(t) = \begin{bmatrix} f_1^+(t) & -g_1^+(t) \\ \tilde{g}_1^+(t) & \tilde{f}_1^+(t) \end{bmatrix} G_1(t) \begin{bmatrix} 0 & g_-(t) \\ -g_-^{-1}(t) & 0 \end{bmatrix}$$

also has a canonical AP factorization.

On the other hand,

$$\begin{aligned} &\begin{bmatrix} f_1^+(t) & -g_1^+(t) \\ \tilde{g}_1^+(t) & \tilde{f}_1^+(t) \end{bmatrix} G_1(t) \begin{bmatrix} 0 & g_-(t) \\ -g_-^{-1}(t) & 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} t^{-n_0} & 0 \\ -\tilde{f}_1^+(t)t^{-n}/g_-(t) & t^{n_0} \end{bmatrix} \end{aligned} \tag{4.12}$$

and since $n_0 > 0$, the matrix-valued function on the right of (4.12) has an AP factorization with AP indices $\pm n_0 \neq 0$. Thus we get a contradiction, and assertion (ii) is proved. ∎

THEOREM 4.2. *Let G be a periodic matrix function defined by* (4.1), *with f_+ and f_k^- defined by* (4.2) *and $t = e^{i\xi}$. Then G has a canonical AP factorization if and only if the roots of the polynomial f_+ do not belong to the set $\{s_k\}_{k=1}^n$.*

Proof. Since the spectrum of the function f given by (4.5) is bounded and the φ transform gives $G_\varphi(t) = G(\varphi t)$, it follows from Lemma 5.2 in [1] that the matrix functions

$$G(t) = \begin{pmatrix} t^n & 0 \\ f(t) & t^{-n} \end{pmatrix}, \qquad G_\varphi(t) = \begin{pmatrix} \varphi^n t^n & 0 \\ f(\varphi t) & \varphi^{-n} t^{-n} \end{pmatrix},$$

$$\tilde{G}_\varphi(t) = \begin{pmatrix} t^n & 0 \\ f(\varphi t) & t^{-n} \end{pmatrix}$$

admit canonical AP factorizations only simultaneously. Choose φ such that $|s_k \varphi^{-1}| < 1$ for all $k = 1, \ldots, n$. It remains to apply Lemma 4.1 to $\tilde{G}_\varphi$. ∎

With the help of Proposition 3.1 from this paper and Corollary 5.4 or Theorem 5.6 taken from [1], it is possible to construct explicit canonical AP factorizations for matrix functions of the form (4.1).

5. AP_W MATRIX FUNCTIONS

The criteria obtained in [2] allow us to identify new classes of AP_W matrix functions that have a canonical AP_W factorization. The next theorem gives one of them.

Again the mentioned criteria give only the sufficient conditions for the canonical AP factorizability of matrix functions considered below. To prove the necessity of these conditions we will use Theorem 2.1 and Lemma 4.1.

THEOREM 5.1. *Let $G \in [AP_W]_{2,2}$ be defined by* (1.1) *with*

$$f(\xi) = \exp(i\gamma\xi)\prod_{k=1}^{n}\left(a_k \exp(i\mu_k\xi) - b_k\right)\Big/\prod_{j=1}^{m}\left(c_j\exp(i\nu_j\xi) - d_j\right), \tag{5.1}$$

where $\gamma \in \mathbb{R}$; $a_k b_k \neq 0$, $k = 1, \dots, n$; $c_j d_j \neq 0$, $j = 1, \dots, m$;

$$|\nu_1| + \cdots + |\nu_m| \leq \lambda, \tag{5.2}$$

and either

$$\gamma \geq 0; \qquad \mu_k > 0, \qquad k = 1, \dots, n; \qquad \nu_j < 0, \qquad j = 1, \dots, m, \tag{5.3}$$

or

$$\gamma \leq 0; \qquad \mu_k < 0, \qquad k = 1, \dots, n; \qquad \nu_j > 0, \qquad j = 1, \dots, m. \tag{5.4}$$

Then the matrix-valued function G has a canonical AP factorization if and only if the following conditions hold:

(i) $|c_j d_j^{-1}| < 1$, $j = 1, \dots, m$;

(ii) $\gamma = 0$ *if* $|\nu_1| + \cdots + |\nu_m| < \lambda$;

(iii) *for each* $k = 1, \dots, n$ *and each* $j = 1, \dots, m$,

$$\left|\frac{b_k}{a_k}\right|^{1/|\mu_k|} \exp\left(\frac{i\left(\arg(b_k/a_k) + 2\pi\alpha\right)}{|\mu_k|}\right) \neq \left|\frac{c_j}{d_j}\right|^{1/|\nu_j|} \exp\left(\frac{i\left(\arg(c_j/d_j) + 2\pi\beta\right)}{|\nu_j|}\right)$$

for all $\alpha, \beta \in \mathbb{N}$ *if* μ_k, ν_j *are both rational*;

$$\left|\frac{b_k}{a_k}\right|^{1/|\mu_k|} \neq \left|\frac{c_j}{d_j}\right|^{1/|\nu_j|}$$

otherwise.

Proof. We consider only the case

$$\gamma \geq 0, \qquad \mu_k > 0, \qquad k = 1, \dots, n, \qquad \nu_j < 0, \qquad j = 1, \dots, m;$$

the other situation is studied analogously.

Let the three conditions (i), (ii), and (iii) be satisfied.

Consider

$$f_-(\xi) = (-1)^m \Big/ \left[d_1 d_2 \cdots d_m \prod_{j=1}^{m} \left(1 - c_j d_j^{-1} \exp(i \nu_j \xi)\right)\right].$$

We have $f_-^{\pm 1} \in \mathrm{AP}_W^-$ from (i).

Rewrite (5.1) in the form

$$f(\xi) = f_+(\xi) e^{i\lambda\xi} / g_+(\xi),$$

where

$$f_+(\xi) = a_1 a_2 \cdots a_n \exp(i\gamma\xi) \prod_{k=1}^{n} \left(\exp(i\mu_k \xi) - a_k^{-1} b_k\right),$$

$$\begin{aligned} g_+(\xi) &= \exp(i\lambda\xi)/f_-(\xi) \\ &= (-1)^m d_1 d_2 \cdots d_m \exp\big(i(\lambda + \nu_1 + \cdots + \nu_m)\xi\big) \\ &\quad \times \prod_{j=1}^{m} \left(\exp\big(i|\nu_j|\xi\big) - c_j d_j^{-1}\right). \end{aligned}$$

Obviously, $f_+ \in \mathrm{AP}_W^+$, $g_+ \in \mathrm{AP}_W^+$ in view of (5.2). Moreover, from (ii) and (iii) we get that

$$\inf_{z \in \Pi^+} \max\left\{|f_+(z)|, |g_+(z)|\right\} > 0. \tag{5.5}$$

Indeed, consider the following functions in H_∞^+:

$$\varphi_0(\xi) = \begin{cases} \exp(i\gamma\xi) \\ 1 \end{cases},$$

$$\psi_0(\xi) = \begin{cases} 1, & \text{if } \lambda + \nu_1 + \cdots + \nu_n = 0, \\ \exp\big(i(\lambda + \nu_1 + \cdots + \nu_n)\xi\big), & \text{if } \lambda + \nu_1 + \cdots + \nu_n > 0; \end{cases}$$

$$\varphi_k(\xi) = a_k \exp(i\mu_k \xi) - b_k \qquad (k = 1, \dots, n),$$

$$\psi_j(\xi) = c_j - d_j \exp(-i\nu_j \xi) \qquad (j = 1, \dots, m).$$

Then

$$f_+ = \varphi_0 \varphi_1 \cdots \varphi_n, \qquad g_+ = \psi_0 \psi_1 \cdots \psi_m,$$

and, by virtue of (ii) and (iii) and of the proof of Lemma 3.3,

$$\inf_{z \in \Pi^+} \max\left\{ |\varphi_k(z)|, |\psi_j(z)| \right\} > 0 \tag{5.6}$$

for every $k = 0, 1, \dots, n$ and every $j = 0, 1, \dots, m$. It remains to observe that

$$\begin{aligned}
&\inf_{z \in \Pi^+} \max\left\{ |\varphi_0(z)\varphi_1(z) \cdots \varphi_n(z)|, |\psi_0(z)\psi_1(z) \cdots \psi_m(z)| \right\} \\
&\qquad \geq \inf_{z \in \Pi^+} \min_{k,j} \max\left\{ |\varphi_k(z)|^{n+1}, |\psi_j(z)|^{m+1} \right\} \\
&\qquad \geq \min\left\{ 1, \min_{k,j} \inf_{z \in \Pi^+} \max\left\{ |\varphi_k(z)|^{n+1}, |\psi_j(z)|^{m+1} \right\} \right\} \\
&\qquad \geq \min\left\{ 1, \left(\min_{k,j} \inf_{z \in \Pi^+} \max\left\{ |\varphi_k(z)|, |\psi_j(z)| \right\} \right)^{\max\{m,n\}+1} \right\},
\end{aligned}$$

which implies (5.5) in view of (5.6).

Then by Theorem 4.2 from [1], the matrix-valued function G has a canonical AP factorization.

Conversely, let the matrix-valued function G admit a canonical AP factorization. Since $G \in [AP_W]_{2,2}$, this factorization is automatically a canonical AP_W factorization [18].

Assume that at least one of the conditions (i) or (ii) is violated. Then a small perturbation of coefficients in (5.1) which preserves (see, e.g., Theorem 2.1) the canonical AP_W factorizability of the perturbed matrix function $\tilde{G} = \begin{pmatrix} e_\lambda & 0 \\ \tilde{f} & e_{-\lambda} \end{pmatrix}$ leads to the fulfillment of (iii) for $\tilde{f}$. So we may suppose that (iii) holds for the original function f. Moreover, we may suppose that $|c_j d_j^{-1}| \neq 1$ for all $j = 1, 2, \dots, m$.

Thus let us assume that (iii) holds and that

$$|c_j d_j^{-1}| \begin{cases} < 1, & \text{for } j \in J_0 \subset \{1, 2, \dots, m\}, \\ > 1, & \text{for } j \in J_1 = \{1, 2, \dots, m\} \setminus J_0. \end{cases}$$

Then

$$\left(\prod_{j=1}^{m} \left(c_j \exp(i \nu_j \xi) - d_j \right) \right)^{-1} = f_0^+(\xi) \exp(i \nu_0 \xi) f_0^-(\xi),$$

where

$$f_0^+(\xi) = 1 \Big/ \prod_{j\in J_1} \big(c_j - d_j \exp(-i\nu_j \xi)\big),$$

$$f_0^-(\xi) = 1 \Big/ \prod_{j\in J_0} \big(c_j \exp(i\nu_j \xi) - d_j\big),$$

and

$$\nu_0 = -\sum_{j\in J_1} \nu_j.$$

We have $(f_0^+)^{\pm 1} \in AP_W^+$, $(f_0^-)^{\pm 1} \in AP_W^-$, and, if $J_1 \neq \emptyset$, $\nu_0 > 0$. Then for f defined by (5.1), we get the representation

$$f(\xi) = f_1^+(\xi) e^{i\lambda\xi} / g_1^+(\xi),$$

where

$$\begin{aligned} f_1^+(\xi) &= f_+(\xi) f_0^+(\xi) \exp(i\nu_0 \xi) \\ &= \exp\left(i\left(\gamma - \sum_{j\in J_1} \nu_j\right)\xi\right) \cdot \prod_{k=1}^{n} \big(a_k \exp(i\mu_k \xi) - b_k\big) \Big\backslash \\ &\qquad \prod_{j\in J_1} \big(c_j - d_j \exp(-i\nu_j \xi)\big)^{-1}, \\ g_1^+(\xi) &= \exp(i\lambda\xi)/f_0^-(\xi) \\ &= \exp\left(i\left(\lambda + \sum_{j\in J_0} \nu_j\right)\xi\right) \prod_{j\in J_0} \big(c_j - d_j \exp(-i\nu_j \xi)\big), \end{aligned}$$

and $f_1^+, g_1^+ \in AP_W^+$.

Let

$$\delta = \min\left\{\gamma - \sum_{j\in J_1} \nu_j, \lambda + \sum_{j\in J_0} \nu_j\right\}.$$

If (i) or (ii) is violated we get that $\delta > 0$. Assume $\delta > 0$ and define

$$f_2^+(\xi) = f_1^+(\xi) e^{-i\delta\xi},$$

$$g_2^+(\xi) = g_1^+(\xi) e^{-i\delta\xi}.$$

Then $f_2^+, g_2^+ \in AP_W^+$ and $f = f_2^+ e_\lambda / g_2^+$. Taking into account (iii) and the relation $(\prod_{j\in J_1} (c_j - d_j \exp(-i\nu_j \xi)))^{\pm 1} \in AP_W^+$ we have

$$\inf_{z\in\Pi^+} \max\{|f_2^+(z)|, |g_2^+(z)|\} > 0.$$

Since $\lambda - \delta \geq 0$ and $(e_{-\lambda+\delta} g_2^+)^{\pm 1} = (f_0^-)^{\mp 1} \in AP_W^-$, Corollary 4.3 from [1] ensures that there exist functions $\tilde{f}_2^+, \tilde{g}_2^+ \in AP_W^+$ such that $f_2^+ \tilde{f}_2^+ + g_2^+ \tilde{g}_2^+ = 1$.

If we have corona solutions $\tilde{f}_2^+, \tilde{g}_2^+ \in AP_W^+$, the canonical AP factorizability of G implies the canonical AP factorizability of G_1 defined by

$$G_1(\xi) = \begin{bmatrix} f_2^+(\xi) & -g_2^+(\xi) \\ \tilde{g}_2^+(\xi) & \tilde{f}_2^+(\xi) \end{bmatrix} \begin{bmatrix} e^{i\lambda\xi} & 0 \\ f_2^+(\xi) e^{i\lambda\xi}/g_2^+(\xi) & e^{-i\lambda\xi} \end{bmatrix} \times \begin{bmatrix} 0 & 1/f_0^-(\xi) \\ -f_0^-(\xi) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\delta\xi} & 0 \\ -\tilde{f}_2^+(\xi) f_0^-(\xi) e^{-i\lambda\xi} & e^{i\delta\xi} \end{bmatrix}.$$

Since $\delta > 0$ and $\tilde{f}_2^+ f_0^- e_{-\lambda} \in AP_W$, there exist functions $h_0^\pm \in AP_W^\pm$ such that

$$\tilde{f}_2^+(\xi) f_0^-(\xi) e^{-i\lambda\xi} = h_0^+(\xi) e^{-i\delta\xi} + h_0^-(\xi) e^{i\delta\xi}.$$

This implies the canonical AP factorizability of

$$G_2(\xi) = \begin{bmatrix} 1 & 0 \\ h_0^+(\xi) & 1 \end{bmatrix} \begin{bmatrix} e^{-i\delta\xi} & 0 \\ -\tilde{f}_2^+(\xi) f_0^-(\xi) e^{-i\lambda\xi} & e^{i\delta\xi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ h_0^-(\xi) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\delta\xi} & 0 \\ 0 & e^{i\delta\xi} \end{bmatrix},$$

which is impossible.

Now let (i) and (ii) hold, but assume (iii) is violated.

Since G defined by (1.1) remains canonically AP_W factorizable under small perturbations of the coefficients and under small perturbations of the exponents due to Theorem 2.1, we can find rational exponents $\tilde{\lambda}, \tilde{\gamma}, \tilde{\mu}_k, \tilde{\nu}_j$, $k = 1, \ldots, n$, $j = 1, \ldots, m$ and coefficients $\tilde{a}_k, \tilde{b}_k, \tilde{c}_k, \tilde{d}_k$, different from zero, in all sufficiently small neighborhoods of the initial exponents and coefficients such that all the conditions of Theorem 5.1 except (iii) are satisfied and at least for one pair (k, j) and $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$,

$$\left| \frac{\tilde{b}_k}{\tilde{a}_k} \right|^{1/|\tilde{\mu}_k|} = \left| \frac{\tilde{c}_j}{\tilde{d}_j} \right|^{1/|\tilde{\nu}_j|}, \qquad \left(\arg \frac{\tilde{b}_k}{\tilde{a}_k} + 2\pi\alpha \right) \Big/ |\tilde{\mu}_k| = \left(\arg \frac{\tilde{c}_j}{\tilde{d}_j} + 2\pi\beta \right) \Big/ |\tilde{\nu}_j|.$$

The perturbed matrix-valued function $\tilde{G}$ can be written in a periodic form (4.1) and we conclude from Lemma 4.1 that $\tilde{G}$ does not admit a canonical AP factorization which contradicts the stability of the canonical AP_W factorizability property under small perturbations of the coefficients and exponents. ∎

COROLLARY 5.2. *Let G be defined by* (1.1) *with*

$$f(\xi) = a_0 \exp(i\gamma\xi) \prod_{k=1}^{n} \left(\exp(i\mu_k \xi) - b_k\right) \times \sum_{(\tau_1, \dots, \tau_m) \in \mathscr{R}} c_1^{\tau_1} c_2^{\tau_2} \cdots c_m^{\tau_m} \exp\big(i(\nu_1\tau_1 + \cdots + \nu_m\tau_m)\xi\big), \quad (5.7)$$

where $\gamma \in \mathbb{R}$, $a_0 \neq 0$, $b_1 b_2 \cdots b_n \neq 0$, $c_1 c_2 \cdots c_m \neq 0$, $|\nu_1| + \cdots + |\nu_m| \leq \lambda$,

$$\mathscr{R} = \big\{(\tau_1, \dots, \tau_m) : |\nu_1|\tau_1 + \cdots + |\nu_m|\tau_m \leq \lambda + |\gamma| + |\mu_1| + \cdots + |\mu_n|, \\ \tau_j \in \{0, 1, \dots\},\ j = 1, \dots, m\big\},$$

and either condition (5.3) *or* (5.4) *of Theorem* 5.1 *is satisfied. Then G has a canonical AP factorization if and only if*

(i) $\gamma = 0$ *in case* $|\nu_1| + \cdots + |\nu_m| < \lambda$,

(ii) *for each* $k = 1, \dots, n$ *and each* $j = 1, \dots, m$,

$$|b_k|^{1/|\mu_k|} \exp\big(i(\arg b_k + 2\pi\alpha)/|\mu_k|\big) \\ \neq |c_j|^{1/|\nu_j|} \exp\big(i(\arg c_j + 2\pi\beta)/|\nu_j|\big) \quad \textit{for all } \alpha, \beta \in \mathbb{N}$$

if μ_k, ν_j *are both rational*;

$$|b_k|^{1/|\mu_k|} \neq |c_j|^{1/|\nu_j|},$$

otherwise.

Proof. Consider G satisfying condition (5.3). Since the spectrum $\Omega(f)$ of the function f given by (5.7) is bounded, it follows from Lemma 5.2 in [1] that the matrix-valued function G has a canonical AP factorization if and only if the matrix function

$$\tilde{G}_\varphi(\xi) = \begin{bmatrix} e^{i\lambda\xi} & 0 \\ f_\varphi(\xi) & e^{-i\lambda\xi} \end{bmatrix}, \qquad \varphi \in \mathbb{C} \setminus \{0\},$$

with

$$f_\varphi(\xi) = a_0 \varphi^\gamma \exp(i\gamma\xi) \prod_{k=1}^{n} \left(\varphi^{\mu_k} \exp(i\mu_k \xi) - b_k\right)$$
$$\times \sum_{(\tau_1, \dots, \tau_m) \in \mathscr{R}} c_1^{\tau_1} c_2^{\tau_2} \cdots c_m^{\tau_m} (\varphi \exp(i\xi))^{\nu_1\tau_1 + \cdots + \nu_m\tau_m}$$

has a canonical AP factorization for every $\varphi \in \mathbb{C} \setminus \{0\}$.

Then we may choose $\varphi > 0$ such that

$$|c_j \varphi^{\nu_j}| < 1 \quad \text{for all } j = 1, 2, \dots, m,$$

and find $h_- \in AP_W^-$ such that

$$\begin{aligned} f_{\varphi,h}(\xi) &= f_\varphi(\xi) + h_-(\xi)\exp(-i\lambda\xi) \\ &= a_0(-1)^m \varphi^\gamma \exp(i\gamma\xi) \\ &\quad \times \prod_{k=1}^{n} \left(\varphi^{\mu_k}\exp(i\mu_k \xi) - b_k\right) \Big/ \prod_{j=1}^{m} \left(c_j \varphi^{\nu_j} \exp(i\nu_j \xi) - 1\right). \end{aligned}$$

Since the matrix functions $\tilde{G}_\varphi$ and $\tilde{G}_{\varphi,h}$ defined by (1.1) with $f(\xi) = f_{\varphi,h}(\xi)$, have canonical AP factorizations only simultaneously, we get from Theorem 5.1 the assertion of the corollary. ∎

6. TRINOMIAL MATRIX FUNCTIONS

In this section we obtain a criterion of canonical AP factorizability and explicit expressions of the AP factors for the matrix-valued function

$$G(\xi) = \begin{bmatrix} e^{i\lambda\xi} & 0 \\ c_{-1}e^{-i\nu\xi} - c_0 + c_1 e^{i\alpha\xi} & e^{-i\lambda\xi} \end{bmatrix}, \tag{6.1}$$

where

$$c_{-1}c_0c_1 \neq 0, \qquad 0 < \alpha, \nu < \lambda = \alpha + \nu. \tag{6.2}$$

A criterion of canonical AP factorization for (6.1) is already known (see [12, Theorem 5.1]), but here we prove it in an essentially simpler way. We emphasize that if G is defined by (6.1) and does not have canonical AP factorization, it is not AP factorizable at all (see [12]).

According to Lemma 3.3 define

$$k_j = \tilde{E}((j+1)\lambda/\alpha), \qquad \tau_j = E((j-1)\lambda/\alpha), \qquad j \in \mathbb{Z}. \quad (6.3)$$

Let also

$$c_{\pm 1,0} = c_{\pm 1} c_0^{-1}, \qquad s = c_{-1} c_1^{-1}.$$

THEOREM 6.1. *Let G be defined by* (6.1) *and* (6.2). *Then the matrix-valued function G has a canonical AP factorization if and only if the following inequalities hold*:

$$c_1^k c_{-1}^m \neq c_0^{k+m} \quad (6.4)$$

if $\beta = \nu/\alpha$ is rational, where $k, m \in \mathbb{N}$ are relatively prime numbers such that $\nu/\alpha = k/m$;

$$|c_1|^{\beta} |c_{-1}| \neq |c_0|^{\beta+1} \quad (6.5)$$

if $\beta = \nu/\alpha$ is irrational.

The factors of a canonical AP factorization $G = G_+ G_-$ have the following form:

(i) *If β is rational and* (6.4) *holds, then*

$$G_+(\xi) = \begin{bmatrix} -\dfrac{c_0^{-1}}{c_{1,0}^{-n} - s^m} \displaystyle\sum_{j=0}^{m-1} \sum_{k=k_{j-1}+1}^{k_j} c_{1,0}^{-n+k} s^j \exp(i(k\alpha - j\lambda)\xi) & \exp(i\lambda\xi) - s \\ -\dfrac{1}{c_{1,0}^{-n} - s^m} \displaystyle\sum_{j=0}^{m-1} c_{1,0}^{-n+k_j+1} s^j \exp\Big(i\big((k_j+1)\alpha - (j+1)\lambda\big)\xi\Big) & c_1 \exp(i\alpha\xi) - c_0 \end{bmatrix},$$

$$G_-(\xi) = \begin{bmatrix} -c_0 s + c_{-1} s \exp(-i\nu\xi) & s\exp(-i\lambda\xi) - 1 \\ 1 - \dfrac{\sum_{j=0}^{m-1} \sum_{k=k_{j-1}+2}^{k_j} c_{1,0}^{-n+k} s^{j+1} e^{i(k\alpha-(j+1)\lambda)\xi}}{c_{1,0}^{-n} - s^m} & -c_0^{-1} \dfrac{\sum_{j=0}^{m-1} \sum_{k=k_{j-1}+1}^{k_j} c_{1,0}^{-n+k} s^j e^{i(k\alpha-(j+1)\lambda)\xi}}{c_{1,0}^{-n} - s^m} \end{bmatrix},$$

where $n, m \in \mathbb{N}$ are co-prime numbers such that $\lambda/\alpha = n/m$.

(ii) *If* β *is irrational and* $|c_1|^{\beta}|c_{-1}| < |c_0|^{\beta+1}$, *then*

$$G_+(\xi) = \begin{bmatrix} -c_0^{-1}\displaystyle\sum_{j=0}^{\infty}\sum_{k=k_{j-1}+1}^{k_j} c_{1,0}^{k} s^j \exp(i(k\alpha - j\lambda)\xi) & \exp(i\lambda\xi) - s \\ -\displaystyle\sum_{j=0}^{\infty} c_{1,0}^{k_j+1} s^j \exp\Big(i\big((k_j+1)\alpha - (j+1)\lambda\big)\xi\Big) & c_1\exp(i\alpha\xi) - c_0 \end{bmatrix}, \tag{6.6}$$

$$G_-(\xi) = \begin{bmatrix} -c_0 s + c_{-1} s\exp(-i\nu\xi) & s\exp(-i\lambda\xi) - 1 \\ 1 - \displaystyle\sum_{j=0}^{\infty}\sum_{k=k_{j-1}+2}^{k_j} c_{1,0}^{k} s^{j+1}\exp(i(k\alpha - (j+1)\lambda)\xi) & -c_0^{-1}\displaystyle\sum_{j=0}^{\infty}\sum_{k=k_{j-1}+1}^{k_j} c_{1,0}^{k} s^{j}\exp(i(k\alpha - (j+1)\lambda)\xi) \end{bmatrix}. \tag{6.7}$$

(iii) *If* β *is irrational and* $|c_1|^{\beta}|c_{-1}| > |c_0|^{\beta+1}$, *then*

$$G_+(\xi) = \begin{bmatrix} c_0^{-1}\displaystyle\sum_{j=1}^{\infty}\sum_{\tau=\tau_j+1}^{\tau_{j+1}} c_{1,0}^{-\tau} s^{-j}\exp(i(-\tau\alpha + j\lambda)\xi) & \exp(i\lambda\xi) - s \\ \displaystyle\sum_{j=1}^{\infty} c_{1,0}^{-\tau_j} s^{-j}\exp\Big(i\big(-\tau_j\alpha + (j-1)\lambda\big)\xi\Big) & c_1\exp(i\alpha\xi) - c_0 \end{bmatrix}, \tag{6.8}$$

$$G_-(\xi) = \begin{bmatrix} -c_0 s + c_{-1} s\exp(-i\nu\xi) & s\exp(-i\lambda\xi) - 1 \\ 1 + \displaystyle\sum_{j=1}^{\infty}\sum_{\tau=\tau_j+1}^{\tau_{j+1}-1} c_{1,0}^{-\tau} s^{-j+1}\exp(i(-\tau\alpha + (j-1)\lambda)\xi) & c_0^{-1}\displaystyle\sum_{j=1}^{\infty}\sum_{\tau=\tau_j+1}^{\tau_{j+1}} c_{1,0}^{-\tau} s^{-j}\exp(i(-\tau\alpha + (j-1)\lambda)\xi) \end{bmatrix}. \tag{6.9}$$

Proof. From Lemma 5.2 in [1] it follows that the matrix-valued function G admits a canonical AP factorization if and only if the matrix function

$$\tilde{G}_\varphi(\xi) = \begin{bmatrix} \exp(i\lambda\xi) & 0 \\ c_{-1}\varphi^{-\nu}\exp(-i\nu\xi) - c_0 + c_1\varphi^{\alpha}\exp(i\alpha\xi) & \exp(-i\lambda\xi) \end{bmatrix},$$

where $\varphi > 0$, has a canonical AP factorization.

Let us choose $\varphi > 0$ such that

$$|c_{-1}c_1^{-1}| < \varphi^{\alpha+\nu}$$

and define

$$\tilde{c}_{-1} = c_{-1}\varphi^{-\nu}, \qquad \tilde{c}_1 = c_1\varphi^{\alpha}.$$

Then $\tilde{G}_\varphi$ coincides with

$$\tilde{G}(\xi) = \begin{bmatrix} e^{i\lambda\xi} & 0 \\ \tilde{c}_{-1}e^{-i\nu\xi} - c_0 + \tilde{c}_1 e^{i\alpha\xi} & e^{-i\lambda\xi} \end{bmatrix},$$

where $|\tilde{c}_{-1}\tilde{c}_1^{-1}| < 1$, and it remains to prove that $\tilde{G}$ has a canonical AP factorization.

Obviously,

$$\tilde{c}_{-1}e^{-i\nu\xi} - c_0 + \tilde{c}_1 e^{i\alpha\xi} = \left(\tilde{c}_1 e^{i\alpha\xi} - c_0\right)\left(\sum_{k=0}^{\infty} s^k e^{-ik\lambda\xi}\right) - h_-(\xi)e^{-i\lambda\xi},$$

where $s = \tilde{c}_{-1}\tilde{c}_1^{-1}$, $|s| < 1$, and

$$h_-(\xi) = \left(-c_0 s + \tilde{c}_1 s^2 e^{-i\nu\xi}\right)\sum_{k=0}^{\infty} s^k e^{-ik\lambda\xi}.$$

Then the function

$$\tilde{f}(\xi) = \tilde{c}_{-1}e^{-i\nu\xi} - c_0 + \tilde{c}_1 e^{i\alpha\xi}$$

is represented in the form

$$\tilde{f} = f_+ f_- - h_- e_{-\lambda},$$

where

$$f_+(\xi) = \tilde{c}_1 e^{i\alpha\xi} - c_0 = \tilde{c}_1\left(e^{i\alpha\xi} - \tilde{c}_1^{-1}c_0\right),$$

$$f_-(\xi) = (1 - se^{-i\lambda\xi})^{-1},$$

$$g_+(\xi) = e^{i\lambda\xi}f_-^{-1}(\xi) = e^{i\lambda\xi} - s.$$

We have $h_- \in AP_W^-$, $f_+ \in AP_W^+(\lambda)$, $f_-^{\pm 1} \in AP_W^-$, and $g_+ \in AP_W^+(\lambda)$. On the other hand,

$$\inf_{z \in \mathbb{C}} \max\{|f_+(z)|, |g_+(z)|\} > 0$$

when

$$\left(\tilde{c}_1^{-1} c_0\right)^n \neq s^m \tag{6.10}$$

if λ/α is rational and n, m are co-prime numbers in $\mathbb{N}$ such that $\lambda/\alpha = n/m$, and

$$|\tilde{c}_1^{-1} c_0|^\lambda \neq |s|^\alpha \tag{6.11}$$

if λ/α is irrational (see the proof of Lemma 3.3).

Note that (6.4) and (6.5) are equivalent to (6.10) and (6.11) because $n - m = k$ and $\lambda/\alpha = \beta + 1$. Then if (6.4) and (6.5) are satisfied, the matrix-valued function $\tilde{G}$ has a canonical AP factorization in view of Theorem 5.6 in [1].

Now let us calculate the factors when G has a canonical AP factorization.

First suppose that $\varphi = 1$, i.e.,

$$\tilde{c}_{\pm 1} = c_{\pm 1}, \qquad s = c_{-1} c_1^{-1}, \quad \text{and} \quad |s| < 1.$$

Since G satisfies the conditions (ii) of Theorem 5.6 [1], the factors $G_\pm$ are given by

$$G_+(\xi) = \begin{bmatrix} \tilde{f}_+(\xi) & g_+(\xi) \\ -\tilde{g}_+(\xi) & f_+(\xi) \end{bmatrix}, \tag{6.12}$$

$$G_-(\xi) = \begin{bmatrix} h_-(\xi)/f_-(\xi) & -1/f_-(\xi) \\ -\tilde{f}_+(\xi) e^{-i\lambda\xi} h_-(\xi) + f_-(\xi) & \tilde{f}_+(\xi) e^{-i\lambda\xi} \end{bmatrix}, \tag{6.13}$$

where $\tilde{f}_+, \tilde{g}_+ \in AP_W^+(\lambda)$ are corona solutions for the corona data $f_+, g_+ \in AP_W^+(\lambda)$.

Let k_j, τ_j be given by (6.3) and $c_{\pm 1, 0} = c_0^{-1} c_{\pm 1}$.

From Lemma 3.3 we get the following corona solutions $\tilde{f}_+, \tilde{g}_+ \in AP_W^+(\lambda)$. If $\lambda/\alpha = n/m$, then

$$\tilde{f}_+(\xi) = -\frac{c_0^{-1}}{c_{1,0}^{-n} - s^m} \sum_{j=0}^{m-1} \sum_{k=k_{j-1}+1}^{k_j} c_{1,0}^{-n+k} s^j \exp(i(k\alpha - j\lambda)\xi),$$

$$\tilde{g}_+(\xi) = \frac{1}{c_{1,0}^{-n} - s^m} \sum_{j=0}^{m-1} c_{1,0}^{-n+k_j+1} s^j \exp\Big(i\big((k_j + 1)\alpha - (j+1)\lambda\big)\xi\Big).$$

If λ/α is irrational and $|c_1|^{\beta}|c_{-1}| < |c_0|^{\beta+1}$, then

$$\tilde{f}_+(\xi) = -c_0^{-1} \sum_{j=0}^{\infty} \sum_{k=k_{j-1}+1}^{k_j} c_{1,0}^k s^j \exp(i(k\alpha - j\lambda)\xi),$$

$$\tilde{g}_+(\xi) = \sum_{j=0}^{\infty} c_{1,0}^{k_j+1} s^j \exp\Big(i\big((k_j+1)\alpha - (j+1)\lambda\big)\xi\Big).$$

If λ/α is irrational and $|c_1|^{\beta}|c_{-1}| > |c_0|^{\beta+1}$, then

$$\tilde{f}_+(\xi) = c_0^{-1} \sum_{j=1}^{\infty} \sum_{\tau=\tau_j+1}^{\tau_{j+1}} c_{1,0}^{-\tau} s^{-j} \exp(i(-\tau\alpha + j\lambda)\xi),$$

$$\tilde{g}_+(\xi) = -\sum_{j=1}^{\infty} c_{1,0}^{-\tau_j} s^{-j} \exp\Big(i\big(-\tau_j\alpha + (j-1)\lambda\big)\xi\Big).$$

This gives G_+ by (6.12). It can be checked directly that

$$\begin{aligned}
&-\tilde{f}_+(\xi)e^{-i\lambda\xi}h_-(\xi) + f_-(\xi) \\
&\quad = \big(1 - \tilde{f}_+(\xi)e^{-i\lambda\xi}h_-(\xi)/f_-(\xi)\big)f_-(\xi) \\
&\quad = \begin{cases}
1 - \dfrac{1}{c_{1,0}^{-n} - s^m} \displaystyle\sum_{j=0}^{m-1} \sum_{k=k_{j-1}+2}^{k_j} c_{1,0}^{-n+k} s^{j+1} \exp(i(k\alpha - (j+1)\lambda)\xi), \\
\quad \text{if } \lambda/\alpha \text{ is rational}, \\
1 - \displaystyle\sum_{j=0}^{\infty} \sum_{k=k_{j-1}+2}^{k_j} c_{1,0}^{k} s^{j+1} \exp(i(k\alpha - (j+1)\lambda)\xi), \\
\quad \text{if } \lambda/\alpha \text{ is irrational and } |c_1|^{\beta}|c_{-1}| < |c_0|^{\beta+1}, \\
1 + \displaystyle\sum_{j=1}^{\infty} \sum_{\tau=\tau_{j+1}}^{\tau_{j+1}-1} c_{1,0}^{-\tau} s^{-j+1} \exp(i(-\tau\alpha + (j-1)\lambda)\xi), \\
\quad \text{if } \lambda/\alpha \text{ is irrational and } |c_1|^{\beta}|c_{-1}| > |c_0|^{\beta+1}.
\end{cases}
\end{aligned}$$

This gives G_- by (6.13).

It remains to observe that the formulas of $G_\pm$ obtained under the assumption $|s| < 1$ are also valid when $|s| \geq 1$. This can be easily checked by direct multiplication.

Let us finally show that the conditions (6.4) and (6.5) are necessary for the canonical AP factorizability of G.

If $\beta = \nu/\alpha$ is rational, the matrix function G has a canonical AP factorization only simultaneously with

$$G_0(\xi) = \begin{bmatrix} e^{in\xi} & 0 \\ c_{-1}e^{-ik\xi} - c_0 + c_1 e^{im\xi} & e^{-in\xi} \end{bmatrix}, \tag{6.14}$$

where $n = k + m$ and $\nu/\alpha = k/m$, because $G_0(\xi) = G(m\xi/\alpha)$.

It follows from Lemma 4.2 in [5] (see also [14]) that the matrix-valued function (6.14) has a canonical AP factorization if and only if

$$\det\left(c_{-1}\delta_{i+k,j} - c_0\delta_{i,j} + c_1\delta_{i,m+j}\right)_{i,j=1}^{n} \neq 0, \tag{6.15}$$

where δ_{ij} is the Kronecker symbol, i.e.,

$$(-1)^n\left(c_0^n - c_1^k c_{-1}^m\right) \neq 0.$$

Thus we have that (6.4) is a necessary condition if $\beta = \nu/\alpha$ is rational.

Necessity of (6.5) in case of irrational $\beta = \nu/\alpha$ is proved in [12]. To keep the exposition self-contained, we reproduce it here.

Let

$$|c_1|^{\beta}|c_{-1}| = |c_0|^{\beta+1}.$$

If G has a canonical AP factorization, then with the help of a small perturbation of c_1, c_{-1} we get a matrix function of the same form for which (6.5) holds. If (6.5) is satisfied, then (6.6)–(6.9) give that

$$d(G) = \begin{cases} \begin{bmatrix} -c_0^{-1} & -s \\ 0 & -c_0 \end{bmatrix}\begin{bmatrix} -c_0 s & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_0^{-1} \\ -c_0 & 0 \end{bmatrix}, \\ \quad \text{if } |c_1|^{\beta}|c_{-1}| < |c_0|^{\beta+1}, \\ \begin{bmatrix} 0 & -s \\ s^{-1} & -c_0 \end{bmatrix}\begin{bmatrix} -c_0 s & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -c_{-1}c_1^{-1} & 0 \\ -2c_0 & -c_{-1}^{-1}c_1 \end{bmatrix}, \\ \quad \text{if } |c_1|^{\beta}|c_{-1}| > |c_0|^{\beta+1}. \end{cases}$$

The mapping $G \mapsto d(G)$ acts continuously from the set of canonically AP factorizable AP_W matrix functions to the set of constant complex matrices (see, e.g., Theorem 7 and Corollary 8 in [8]). Since for the perturbed matrix functions considered we may obtain both possibilities for $d(G)$, we get a contradiction. ∎

Let us remark again that in Theorem 5.1 of [12] it is proved that G, defined by (6.1), is not AP factorable at all if $|c_1|^{\beta}|c_{-1}| \neq |c_0|^{1+\beta}$ and $\beta = \nu/\alpha$ is irrational. Hence this condition gives not only a criterion of canonical AP factorizability, but also a criterion of AP factorizability with any partial AP indices. In the case of rational β, G is always AP factorizable, but it can happen (see, e.g., Theorems 3.1 and 3.2 in [13]) that G is AP factorizable with nonzero partial AP indices if (6.4) is violated.

Finally, note that with the help of Theorem 6.1 we may construct explicit canonical AP factorizations for all triangular 2×2 matrix functions considered in [2].

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